18.05 Lecture 17
March 16, 2005

Properties of Expectation. Law of Large Numbers.
$\mathbb{E}(X_1 + ... + X_n) = \mathbb{E}X_1 + ... + \mathbb{E}X_n$

Matching Problem (n envelopes, n letters)
Expected number of letters in correct envelopes?
Y - number of matches
$X_i = \{1$, letter i matches; 0, otherwise$\}$, Y $= X_1 + ... + X_n$
$\mathbb{E}(Y) = \mathbb{E}X_1 + ... + \mathbb{E}X_n$, but $\mathbb{E}X_i = 1 \times \mathbb{P}(X_i = 1) + 0 \times \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{n}$
Therefore, expected match = 1:

$$\mathbb{E}(Y) = n \times \frac{1}{n} = 1$$

If $X_1, ..., X_n$ are independent, then $\mathbb{E}(X_1 \times ... \times X_n) = \mathbb{E}X_1 \times ... \times \mathbb{E}X_n$
As with the sum property, we will prove for two variables:
$\mathbb{E}X_1X_2 = \mathbb{E}X_1 \times \mathbb{E}X_2$
joint p.f. or p.d.f.: $f(x_1, x_2) = f_1(x_1)f_2(x_2)$
$\mathbb{E}X_1X_2 = \int\int x_1x_2f(x_1, x_2)dx_1dx_2 = \int\int x_1x_2f_1(x_1)f_2(x_2)dx_1dx_2 =$
$= \int f_1(x_1)x_1dx_1 \int f_2(x_2)x_2dx_2 = \mathbb{E}X_1 \times \mathbb{E}X_2$

X_1, X_2, X_3 - independent, uniform on [0, 1]. Find $\mathbb{E}X_1^2(X_2 - X_3)^2$.
$= \mathbb{E}X_1^2\mathbb{E}(X_2 - X_3)^2$ by independence.
$= \mathbb{E}X_1^2\mathbb{E}(X_2^2 - 2X_2X_3 + X_3^2) = \mathbb{E}X_1^2(\mathbb{E}X_2^2 + \mathbb{E}X_3^2 - 2\mathbb{E}X_2X_3)$
By independence of $X_2, X_3; = \mathbb{E}X_1^2(\mathbb{E}X_2^2 + \mathbb{E}X_3^2 - 2\mathbb{E}X_2\mathbb{E}X_3)$
$\mathbb{E}X_1 = \int_0^1 x \times 1dx = 1/2, \mathbb{E}X_1^2 = \int_0^1 x^2 \times 1dx = 1/3$ (same for X_2 and X_3)
$\mathbb{E}X_1^2(X_2 - X_3)^2 = \frac{1}{3}(\frac{1}{3} + \frac{1}{3} - 2(\frac{1}{2})(\frac{1}{2})) = \frac{1}{18}$

For discrete random variables, X takes values 0, 1, 2, 3, ...
$\mathbb{E}(X) = \sum_{n=0}^{\infty} n\mathbb{P}(x = n)$
for n = 0, contribution = 0; for n = 1, $\mathbb{P}(1)$; for n = 2, $2\mathbb{P}(2)$; for n = 3, $3\mathbb{P}(3)$; ...
$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$

Example: X - number of trials until success.
$\mathbb{P}(success) = p$
$\mathbb{P}(failure) = 1 - p = q$

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n) = \sum_{n=1}^{\infty}(1-p)^{n-1} = 1 + q + q^2 + ... = \frac{1}{1-q} = \frac{1}{p}$$

Formula based upon reasoning that the first n - 1 times resulted in failure.
Much easier than the original formula:
$\sum_{n=0}^{\infty} n\mathbb{P}(X = n) = \sum_{n=1}^{\infty} n(1-p)^{n-1}p$

Variance:

Definition: Var$(X) = \mathbb{E}(X - \mathbb{E}(X))^2 = \sigma^2(X)$
Measure of the deviation from the expectation (mean).
Var$(X) = \int(X - \mathbb{E}(X))^2 f(x)dx$ - moment of inertia.

$\sim \sum(X - \text{ center of gravity })^2 \times m_x$

Standard Deviation:

$\sigma(X) = \sqrt{\text{Var}(X)}$
$\text{Var}(aX + b) = a^2\text{Var}(X)$
$\sigma(aX + b) = |a|\sigma(X)$

Proof by definition:
$\mathbb{E}((aX + b) - \mathbb{E}(aX + b))^2 = \mathbb{E}(aX + b - a\mathbb{E}(X) - b)^2 = a^2\mathbb{E}(X - \mathbb{E}(X))^2 = a^2\text{Var}(X)$

Property: $\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}(X))^2$
Proof:
$\text{Var}(X) = \mathbb{E}(X - \mathbb{E}(X))^2 = \mathbb{E}(X^2 - 2X\mathbb{E}(X) + (\mathbb{E}(X))^2) =$
$\mathbb{E}X^2 - 2\mathbb{E}(X) \times \mathbb{E}(X) + (\mathbb{E}(X))^2 = \mathbb{E}(X)^2 - (\mathbb{E}(X))^2$

Example: $X \sim U[0, 1]$

$$\mathbb{E}X = \int_0^1 x \times 1dx = \frac{1}{2}, \mathbb{E}X^2 = \int_0^1 x^2 \times 1dx = \frac{1}{3}$$

$$\text{Var}(X) = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}$$

If $X_1, ..., X_n$ are independent, then $\text{Var}(X_1 + ... + X_n) = \text{Var}(X_1) + ... + \text{Var}(X_n)$
Proof:

$$\text{Var}(X_1 + X_2) = \mathbb{E}(X_1 + X_2 - \mathbb{E}(X_1 + X_2))^2 = \mathbb{E}((X_1 - \mathbb{E}X_1) + (X_2 - \mathbb{E}X_2))^2 =$$

$$= \mathbb{E}(X_1 - \mathbb{E}X_1)^2 + \mathbb{E}(X_2 - \mathbb{E}X_2)^2 + 2\mathbb{E}(X_1 - \mathbb{E}X_1)(X_2 - \mathbb{E}X_2) =$$

$$= \text{Var}(X_1) + \text{Var}(X_2) + 2\mathbb{E}(X_1 - \mathbb{E}X_1) \times \mathbb{E}(X_2 - \mathbb{E}X_2)$$

By independence of X_1 and X_2:

$$= \text{Var}(X_1) + \text{Var}(X_2)$$

Property: $\text{Var}(a_1X_1 + ... + a_nX_n + b) = a_1^2\text{Var}(X_1) + ... + a_n^2\text{Var}(X_n)$

Example: Binomial distribution - $B(n, p), \mathbb{P}(X = k) = \binom{n}{k}p^k(1 - p)^{n-k}$
$X = X_1 + ... + X_n, X_i = \{1, \text{ Trial i is success }; 0, \text{ Trial i is failure.}\}$
$\text{Var}(X) = \sum_{i=1}^n \text{Var}(X_i)$
$\text{Var}(X_i) = \mathbb{E}X_i^2 - (\mathbb{E}X_i)^2, \mathbb{E}X_i = 1(p) + 0(1 - p) = p; \mathbb{E}X_i^2 = 1^2(p) + 0^2(1 - p) = p.$
$\text{Var}(X_i) = p - p^2 = p(1 - p)$
$\text{Var}(X) = np(1 - p) = npq$

Law of Large Numbers:
$X_1, X_2, ..., X_n$ - independent, identically distributed.

$$S_n = \frac{X_1 + ... + X_n}{n} \xrightarrow{n \to \infty} \mathbb{E}X_1$$

Take $\epsilon > 0$ - but small, $\mathbb{P}(|S_n - \mathbb{E}X_1| > \epsilon) \to 0$ as $n \to \infty$
By Chebyshev's Inequality:

$$\mathbb{P}((S_n - \mathbb{E}X_1)^2 > \epsilon^2) = \mathbb{P}(Y > M) \leq \frac{1}{M}\mathbb{E}Y =$$

$$\frac{1}{\epsilon^2}\mathbb{E}(S_n - \mathbb{E}X_1)^2 = \frac{1}{\epsilon^2}\mathbb{E}(\frac{X_1 + ... + X_n}{n} - \mathbb{E}X_1)^2 = \frac{1}{\epsilon^2}\mathrm{Var}(\frac{1}{n}(X_1 + ... + X_n)) =$$

$$= \frac{1}{\epsilon^2 n^2}(\mathrm{Var}(X_1) + ... + \mathrm{Var}(X_n)) = \frac{n\mathrm{Var}(X_1)}{\epsilon^2 n^2} = \frac{\mathrm{Var}(X_1)}{n\epsilon^2} \to 0$$

for large n.

** End of Lecture 17